Sub-mfd

Ref. Mean curvature flow in higher co-dimension
- intro. and survey
By Smoczyk.

Section 2-3.

Recall: Given an ambient mfd (N^m, h),

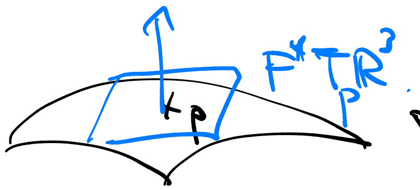
M^n is an embedded sub-mfd if $\exists F: M^n \to N^m$ $(n \leq m)$

s.t. $F(M) \cong M$.

pull-back bundle. $=$

$$\begin{tikzcd} F^*TN \arrow[d] & TN \arrow[d] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

eg:



$\Sigma^2 \subseteq R^3$ with $F =$ inclusion map

$T_pR^3 =$ usual vector bundle.

$F^*T_pR^3 =$ tangent space at p $=$ tangent plane to Σ $\oplus$ normal v.s. at p.

Then the metric h on TN induce the connection

$\overline{\nabla}$ on F^*TN by

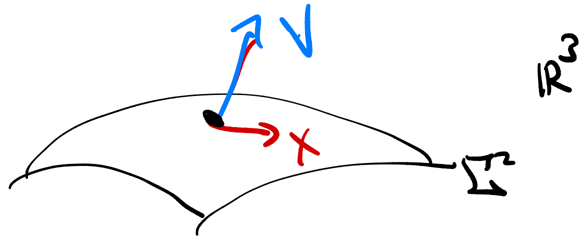
$\forall \ X \in \Gamma(TM), \ V \in \Gamma(F^*TN).$

$$\bar{\nabla}_X V \triangleq \tilde{\nabla}_{dF(X)} V$$ where $\tilde{\nabla}$ = levi-civita connection of (N, h).

locally, $$\boxed{\bar{\nabla}_i V^\mu = \partial_i V^\mu + F^\alpha_i \tilde{\Gamma}^\mu_{\alpha\beta} V^\beta.}$$

(similar to D in parallel transport)

eg:



- Keep in mind: $V \in TM$ as $dF(V) \in$ "F^*TN" $\subseteq TN$"

$T^\perp_p M = \{ \nu \in T_{F(p)} N \mid h(\nu, dF(u)) = 0, \ \forall u \in T_p M \}$

$$T^\perp M = \coprod_{p \in M} T^\perp_p M \quad \text{(normal bundle)}$$

$$\Rightarrow \boxed{T_{F(p)} N \underset{\substack{\uparrow \\ \text{based on } h.}}{\cong} T^\perp_p M \oplus dF|_p (T_p M).}$$

Q: How to define connection on M based on embedding?

Approach 1: $M \xrightarrow{F} N$

$\Rightarrow$ induce g on M by $\underset{\substack{\parallel \\ \text{Riemannian metric on } M.}}{g} \triangleq F^* h$

$\Rightarrow$ induce a Levi-Civita connection ∇^1 wrt g on M.

Approach 2: define for any $X, Y \in \Gamma(TM)$,

$$dF(\nabla^2_X Y) = \Big(\tilde{\nabla}_{dF(X)} \overline{dF(Y)}\Big)^T \in dF(TM).$$

Some smooth extension of $dF(Y)$ to $\Gamma(TN)$

Q1: Why is ∇^2 well-defined??

Q2: Are ∇^1 and ∇^2 equivalent??

Q1: Yes!! i.e. Given $X, Y \in \Gamma(TM)$, if $\tilde{Y}, \overline{Y} \in \Gamma(TN)$ are two smooth extension of $dF|_p(Y)$, Then

$$\tilde{\nabla}_{dF(X)} \tilde{Y} = \tilde{\nabla}_{dF(X)} \overline{Y} \quad \text{at } F(p).$$

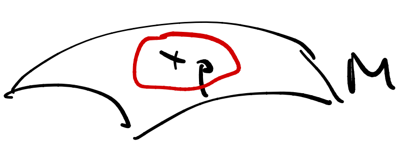
pf: Suffices to show that if $Y \equiv 0$ on M, then

$$\tilde{\nabla}_{dF(X)} \tilde{Y} = 0 \quad \forall \text{ extension } \tilde{Y} \text{ of } dF(Y).$$

$\because F =$ embedding

$\therefore$ Might assume locally $F =$ inclusion. and $\{x^1, \dots, x^n\}$ coordinate of $M \subset N$ at p

and $\{x^1, \cdots, x^n, x^{n+1}, \cdots, x^m\}$ coordinate of N at p.



Suffices to consider

- $X = \partial_i$, $i = 1, 2, \cdots, n$
- $\tilde{Y} = \sum_{j=1}^{m} \varphi_j \partial_j$ s.t.

  $\varphi_j = 0$ on $\{x^{n+1} = \cdots = x^m = 0\}$

$$\tilde{\nabla}_X \tilde{Y} = \tilde{\nabla}_{\partial_i}(\varphi_j \partial_j) = \partial_i \varphi_j \cdot \partial_j + \varphi_j \tilde{\nabla}_{\partial_i} \partial_j$$

$$\underset{\text{at } p \in M}{=} \overset{0 \ \because\ \varphi_j = 0 \text{ on } M}{\cancel{\partial_i \varphi_j}} \cdot \partial_j + \overset{0}{\cancel{\varphi_j}} \cdot \Gamma_{ij}^k \partial_k = 0 \ \#$$

Q2: Yes $\nabla^1 = \nabla^2$ on TM.

∵ Levi-Civita connection is unique connection s.t.

① $\nabla g = 0$ ② $T^{\nabla} = 0$.

∴ suffices to check $\nabla^2 g = 0$, $T^{\nabla^2} = 0$.

$$\boxed{dF([X, Y]) = [dF(X), dF(Y)] \text{ by local computation}}$$

②: ∵ F = embedding.

∴ suffices to show $dF(\nabla^2_X Y - \nabla^2_Y X - [X, Y]) = 0$.

$\forall\ X, Y \in \Gamma(TM)$,

$$dF\left(\nabla^2_X Y - \nabla^2_Y X - [X,Y]\right)$$

$$= \left(\tilde{\nabla}_{dF(X)} dF(Y) - \tilde{\nabla}_{dF(Y)} dF(X)\right)^T - [dF(X), dF(Y)].$$

$$= [dF(X), dF(Y)]^T - [dF(X), dF(Y)]$$

$$= 0 \quad \boxed{\because\ dF[X,Y] \in dF(TM).} \quad \#.$$

$\therefore\ \nabla^2 =$ torsion free!!

①: $\forall\ Z, X, Y \in \Gamma(TM)$, $p \in M$

$$Z\left(g(X,Y)\right)\Big|_p = Z\left(h(dF(X), dF(Y))\right)\Big|_p$$

$$= dF(Z)\left(h(dF(X), dF(Y))\right)\Big|_{F(p)},$$

$$\overset{\boxed{\tilde{\nabla} h = 0}}{=} h\left(\left(\tilde{\nabla}_{dF(Z)} dF(X)\right)^T, \underline{\underline{dF(Y)}}\right)$$

$$+\ h\left(\underline{\underline{dF(X)}}, \left(\tilde{\nabla}_{dF(Z)} dF(Y)\right)^T\right)$$

$$= h\left(dF(\nabla^2_Z X), dF(Y)\right)$$

$$+\ h\left(dF(X), dF(\nabla^2_Z Y)\right)$$

$$= g(\nabla^2_Z X, Y) + g(X, \nabla^2_Z Y) \quad \#$$

$$\therefore \nabla^2 g = 0. \quad \#.$$

$$\begin{cases} \nabla^2 g = 0 \\ T(\nabla^2) = 0 \end{cases} \Rightarrow \nabla^1 = \nabla^2$$

Likewise, might define $\nabla^\perp$ on $T^\perp M$ by

$$\nabla^\perp_X \nu \triangleq (\nabla_X \nu)^\perp \quad \text{for } X \in \Gamma(TM), \ \nu \in \Gamma(T^\perp M).$$

Extend ∇, $\bar{\nabla}$ naturally to all. (By duality)

$$\Gamma(TM^{\otimes k} \otimes T^*M^{\otimes l} \otimes F^*TN^{\otimes r} \otimes F^*T^*N^{\otimes s}).$$

Call them : $\tilde{\nabla}$.

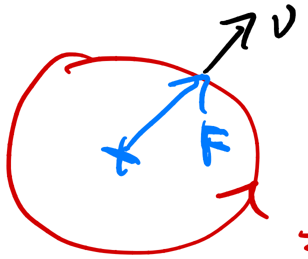
- **Defn** : For $X, Y \in \Gamma(TM)$, define

$$A(X,Y) \triangleq \left(\tilde{\nabla}_{dF(X)} dF(Y)\right)^\perp \in T^\perp M \subseteq F^*TN$$

We call A to be 2nd fundamental form.

Rmk : Some ppl define $A \triangleq -(\tilde{\nabla}_{dF(X)} dF(Y))^\perp$

so that.



$S^2 \subseteq \mathbb{R}^3.$

$\Rightarrow \langle -\tilde{\nabla}_{dF(X)} dF(Y), \nu \rangle = g(X,Y).$

$\Gamma(T^*M \otimes \quad \mathrm{Sym}(T^*M \otimes T^*M))$

prop: $A \in \Gamma(F^*TN \otimes T^*M \otimes T^*M)$

and $A(X,Y) = A(Y,X).$

pf: symmetric: $\forall X,Y \in \Gamma(TM),$

$$A(X,Y) = \left(\tilde{\nabla}_{dF(X)} dF(Y)\right)^{\perp}$$

$$= \tilde{\nabla}_{dF(X)} dF(Y) - \left(\tilde{\nabla}_{dF(X)} dF(Y)\right)^{T}$$

$$= \tilde{\nabla}_{dF(X)} dF(Y) - dF(\nabla_X Y).$$

$$A(Y,X) = \tilde{\nabla}_{dF(Y)} dF(X) - dF(\nabla_Y X)$$

$$\Rightarrow A(X,Y) - A(Y,X)$$

$$= [dF(X), dF(Y)] - dF([X,Y]) = 0 \ \#$$

linear over $C^\infty(M)$:

$$A(fX,Y) = \tilde{\nabla}_{dF(fX)} dF(Y) - dF(\nabla_{fX} Y)$$

$$= \tilde{\nabla}_{f\,dF(X)} dF(Y) - dF(\nabla_{fX} Y)$$

$$= f\left(\tilde{\nabla}_{dF(X)} dF(Y) - dF(\nabla_X Y)\right).$$

$$= f \cdot A(X,Y) \quad \#$$

symmetry $\Rightarrow$ $A(X, fY) = f A(X,Y) \quad \forall X, Y \in \Gamma(TM)$
$f \in C^\infty(M) \quad \#$

locally, $A = A_{ij}^{\alpha}\, dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha}$

where $\{x^i\}$ = local coordinate of M.
$\{y^\alpha\}$ = local coordinate of N.

Recall: $dF \in \Gamma(F^* TN \otimes T^* M)$

locally, $dF = F_i^\alpha\, dx^i \otimes \frac{\partial}{\partial y^\alpha}$

$$F_i^\alpha = \partial_i F^\alpha.$$

prop: $A(X,Y) = (\nabla dF)(X,Y) \quad \forall X,Y \in \Gamma(TM).$

$$\left(\text{i.e. } A^\alpha_{ij} = F^\alpha_{ij} = \partial_i\partial_j F^\alpha - \Gamma^k_{ij} F^\alpha_k + \tilde{\Gamma}^\alpha_{\beta\gamma} F^\beta_i F^\gamma_j . \right)$$

pf: $(\nabla dF)(X,Y) \quad \forall X,Y \in \Gamma(TM).$

$$= (\nabla_X dF)(Y)$$

$$= \nabla_X\big(\underline{dF(Y)}\big) - dF(\nabla_X Y).$$

↓ based on arbitrary extension

$$= \tilde{\nabla}_{dF(X)} dF(Y) - \left(\tilde{\nabla}_{dF(X)} dF(Y)\right)^T$$

$$= \left(\tilde{\nabla}_{dF(X)} dF(Y)\right)^\perp = A(X,Y) \quad \#.$$

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eg: $M = S^2 \subseteq \mathbb{R}^3 = N.$

compute A^ν_{ij} at $p \in S^2$.

Rmk: $\because M^n =$ hypersurface of N^{n+1}

$A_{ij} \in T^\perp M =$ rank 1.

$\boxed{A^\nu_{ij}} = \langle A_{ij}, \nu \rangle$, $\nu =$ unit outward normal.

In Fact, $A^{\nu} \in \Gamma(T^*M \otimes T^*M)$

$A^{\nu} = A^{\nu}_{ij}\, dx^i \otimes dx^j$. locally.

By Rotation, might assume $p = (0,0,1) \in S^2 \subseteq \mathbb{R}^3$.

Direct parametrization : (Might try using spherical coordinate)

$$F(x,y) = (x, y, \sqrt{1-x^2-y^2}) \quad \text{on} \quad \underset{D}{\underset{\|}{\{|x|^2+|y|^2<\varepsilon\}}} \subseteq \mathbb{R}^2.$$

at $p \Leftrightarrow$ at $(0,0) \in D$.

1st fundament form (OR equivalently the induced metric $g = F^*h$) :

At p, $g = g_{ij}\, dx^i \otimes dx^j$

$$= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \quad \text{where}$$

$$F_*\partial_1 = \left(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}\right) = \partial_1 \in T_p\mathbb{R}^3$$

$$F_*\partial_2 = \left(0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}\right) = \partial_2 \in T_p\mathbb{R}^3$$

$$\Rightarrow \begin{cases} g_{11} = (F^*h)_{11} = \langle F_*\partial_1, F_*\partial_1) = 1. \\ g_{12} = 0, \quad g_{22} = 1 \end{cases}$$

$\therefore\ g(p) = I_2 \quad , \quad g^{-1}(p) = I_2 .$

- $F^1 = x \ ; \ F^2 = y \ ; \ F^3 = \sqrt{1-x^2-y^2}$

- At p, $\quad \nu = (\underline{0}, \underline{0}, \underline{1})$

$$A^\nu_{ij} = \langle A_{ij}, \nu \rangle = A^1_{ij} \cdot 0 + A^2_{ij} \cdot 0 + A^3_{ij} \cdot 1$$

$$= A^3_{ij} = F^3_{ij}$$

$$= \partial_i \partial_j F^3 - \Gamma^k_{ij} \overset{0}{\cancel{F^3_k}} + \overset{0}{\cancel{\Gamma^3_{\alpha\beta}}} F^\alpha_i F^\beta_j$$

(By std coord. of $\mathbb{R}^3$.)

(non-zero if we use spherical coord.)

- $F^3_1 = \frac{\partial}{\partial x}\left(\sqrt{1-x^2-y^2}\right)$

$$= \frac{-x}{\sqrt{1-x^2-y^2}} = 0 \quad \text{at } p = F(0,0).$$

- $F^3_2 = 0$ similarly.

$$\Rightarrow A^\nu_{ij} = \partial_i \partial_j F^3 = \partial_i \partial_j \left(\sqrt{1-x^2-y^2}\right)$$

Hence, $A^3_{11} = \partial_x\left(\frac{-x}{\sqrt{1-x^2-y^2}}\right) = \frac{-1}{\sqrt{1-x^2-y^2}} + O(x)$

$$= -1 \quad \text{at} \quad p = F(0,0).$$

$$A^3_{22} = \partial_y \left(\frac{-y}{\sqrt{1-x^2-y^2}} \right) = -1 \quad \text{at } p = F(0,0)$$

$$A^3_{12} = \partial_y \left(\frac{-x}{\sqrt{1-x^2-y^2}} \right) = 0 \quad \text{at } p = F(0,0).$$

$$\boxed{\overset{\text{tensor.}}{A^{\nu}_{\underline{ij}}} = -\delta_{ij} = \underset{\text{convention:}}{-}\,\overset{\text{tensor}}{\underline{g_{ij}}} \quad \text{on } S^2. \quad \#.}$$

(Exercise: Try using $F(\theta,\varphi) = (1,\theta,\varphi)$ under spherical coordinate.)

Now, $(M,g) \xhookrightarrow{F} (N,h)$.

<u>Q</u>: How is $Rm(g)$ Related to $\widetilde{Rm}(h)$??

<u>Gauss equation</u>: $\forall X, Y, Z, W \in \Gamma(TM)$,

$$R(X,Y,Z,W) = \tilde{R}(dF(X), dF(Y), dF(Z), dF(W))$$
$$- \langle A(X,Z), A(Y,W) \rangle + \langle A(X,W), A(Y,Z) \rangle.$$

<u>pf:</u> Suffices to show the gauss formula for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$, $W = \partial_l$ where

$\{\partial_i\}$ is normal coordinate at $p \in M$.

$$R_{ijkl} = \langle \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k , \partial_l \rangle_g \quad \text{at } p.$$

$$\overset{\text{(Normal coord.)}}{=} \partial_i \langle \nabla_{\partial_j} \partial_k , \partial_l \rangle_g - \partial_j \langle \nabla_{\partial_i} \partial_k , \partial_l \rangle_g$$

$$= dF(\partial_i) \cdot \langle \left(\tilde{\nabla}_{dF(\partial_j)} dF(\partial_k) \right)^{\cancel{T}} , dF(\partial_l) \rangle_h$$

$$- dF(\partial_j) \cdot \langle \left(\tilde{\nabla}_{dF(\partial_i)} dF(\partial_k) \right)^{\cancel{T}} , dF(\partial_l) \rangle_h$$

$$= \underline{\langle \left(\tilde{\nabla}_{dF(\partial_i)} \tilde{\nabla}_{dF(\partial_j)} - \tilde{\nabla}_{dF(\partial_j)} \tilde{\nabla}_{dF(\partial_i)} \right) dF(\partial_k) , dF(\partial_l) \rangle_h}$$

$$+ \langle \tilde{\nabla}_{dF(\partial_j)} dF(\partial_k) , \tilde{\nabla}_{dF(\partial_i)} dF(\partial_l) \rangle_h$$

$$- \langle \tilde{\nabla}_{dF(\partial_i)} dF(\partial_k) , \tilde{\nabla}_{dF(\partial_j)} dF(\partial_l) \rangle_h$$

$$= \tilde{R}(dF(\partial_i) , dF(\partial_j) , dF(\partial_k) , dF(\partial_l))$$

$$+ \langle \tilde{\nabla}_{dF(\partial_j)} dF(\partial_k) , \tilde{\nabla}_{dF(\partial_i)} dF(\partial_l) \rangle_h$$

$$- \langle \tilde{\nabla}_{dF(\partial_i)} dF(\partial_k) , \tilde{\nabla}_{dF(\partial_j)} dF(\partial_l) \rangle_h$$

Remain to consider

$$\langle \tilde\nabla_{dF(\partial_j)} dF(\partial_k), \tilde\nabla_{dF(\partial_i)} dF(\partial_l) \rangle.$$

(normal at p)

$$= \langle A(\partial_j, \partial_k) + dF(\overset{0}{\cancel{\nabla_{\partial_j}\partial_k}}), A(\partial_i, \partial_l) + dF(\overset{0}{\cancel{\nabla_{\partial_i}\partial_l}}) \rangle$$

$$= \langle A(\partial_j, \partial_k), A(\partial_i, \partial_l) \rangle_h \ \#$$

Defn: Mean curvature vector

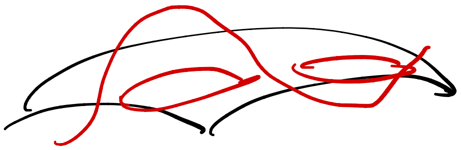
$$\vec H = g^{ij} \vec F_{ij} = \mathrm{tr}_g(A) \in \Gamma(T^\perp M)$$

locally $H = H^\alpha \frac{\partial}{\partial y^\alpha} \in \Gamma(T^\perp M)$.

Why care ??

prop: Suppose $M \xhookrightarrow{F_0} N$ 3 embedded closed sub-mfd. If F_0 = minimal in the sense that for any smooth variation of F_0

Area(M) wrt F_0 = minimal, then $\vec H = 0$.



$\Sigma \subseteq \mathbb{R}^3$

pf: Assumption $\Rightarrow$ $\forall$ $F(t) : M \to N$ s.t.

$$\begin{cases} F(0) = F_0 \\ \partial_t F\big|_{t=0} = T \in \Gamma(F^* TN). \end{cases}$$

then $\partial_t \operatorname{Area}(M, F_t^* h) = 0$ at $t=0$.

$(F^* h)_{ij}$ = time evolving metric on M.

given by

$$F_i^\alpha F_j^\beta \overset{\text{time indep}}{h_{\alpha\beta}} = g_{ij}(t).$$

$$\partial_t \operatorname{Area} = \frac{d}{dt} \int_M d\mu_t = \int_M \partial_t \, d\mu_t.$$

$$= \int_M \frac{1}{2} \operatorname{tr}_g(\partial_t g) \, d\mu_t \quad \text{(Jacobi-formula)}$$

where

$$\operatorname{tr}_g(\partial_t g) = g^{ij}(\partial_t g_{ij}) = g^{ij}(F_i^\alpha F_j^\beta h_{\alpha\beta})'$$

$$= g^{ij}(F_{it}^\alpha F_j^\beta + F_i^\alpha F_{jt}^\beta) h_{\alpha\beta}$$

$$= 2 g^{ij}(T_i^\alpha F_j^\beta) h_{\alpha\beta} \quad (\because g_{ij} = g_{ji})$$

Here, $T_i^\alpha = \nabla_i T^\alpha$.

$$\Rightarrow \quad \partial_t\Big|_{t=0} \text{Area}(M, g(t)) = \int_M \Big(g^{ij}\, \overbrace{T^\alpha_i}^{C^\infty(M)}\, F^\beta_j\, h_{\alpha\beta}\Big) \cdot d\mu_{g_0}.$$

verify $\tilde\nabla_i h_{\alpha\beta} = 0$;

$\parallel$

$\partial_i h_{\alpha\beta} - \tilde\Gamma^\sigma_{\gamma\alpha} h_{\sigma\beta} F^\gamma_i - \tilde\Gamma^\sigma_{\gamma\beta} h_{\alpha\sigma} F^\gamma_i$

$= \tilde\nabla_\gamma h_{\alpha\beta} \cdot F^\gamma_i = 0$

$$\xrightarrow{\text{div. thm}} \; = -\int_M T^\alpha \cdot F^\beta_{ij}\, g^{ij}\, h_{\alpha\beta}\, d\mu_0$$

$$= -\int_M \langle \vec T, \vec H \rangle \, d\mu_t \Big|_{t=0}$$

$$\forall\, T \in \Gamma(F^*TN)$$

$$\Rightarrow \boxed{\vec H = 0 \text{ at } t=0 \text{ on } M}$$

What about 2nd variation??

- Assume $M \xhookrightarrow{F_0} N$ is a hypersurface which is minimal $(H = \langle \vec H, \nu \rangle = 0)$

  $\uparrow$ mean curvature $\in C^\infty(M)$

- Assume variational vector field $= T = f\nu$ where $f \in C^\infty(M)$, $\nu =$ outward normal unit vector.

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(M, g(t)) = \frac{d}{dt}\Big|_{t=0} \int_M -\langle \vec T, \vec H \rangle \, d\mu_t$$

$$= \frac{d}{dt}\Big|_{t=0} \int_M -fH \, d\mu_t$$

$$= \int_M -f H' \, d\mu_t \quad (\because H = 0 \text{ at } t=0.)$$

$$H' = \partial_t (g^{ij} A_{ij}) = \partial_t g^{ij} \cdot A_{ij} + g^{ij} \partial_t A_{ij}$$

$$= I + II.$$

$$I = -2 g^{il} g^{kj} (F^\beta_{lt} F^\gamma_k) h_{\beta\gamma} A_{ij} \qquad \left(\begin{array}{l} A_{ij} = A_{ji} \\ g^{ij} = g^{ji} \end{array} \right)$$

$$= -2 g^{il} g^{kj} F^\gamma_k h_{\beta\gamma} A_{ij} (f_l \nu^\beta + f \nu^\beta_l)$$

$$F^\beta_{tl} = F^\beta_{lt} = (f \nu^\beta)_l .$$

$$= -2f \, g^{il} g^{kj} F^\gamma_k h_{\beta\gamma} A_{ij} \underline{\nu^\beta_l} . \quad (\because \nu \perp dF(TM))$$

Moreover, $\nu_l = \sum_{\beta=1}^{n} \nu_l^\beta \cdot \frac{\partial}{\partial y^\beta}$.

and $\langle \nu_l , \nu \rangle = \frac{1}{2} \partial_l \langle \nu, \nu \rangle = 0$

$$\Rightarrow \quad \nu_l \in dF(TM).$$

As $\langle \nu_l , dF(\partial_i) \rangle = - \langle \nu , \nabla_l dF(\partial_i) \rangle = - A_{il}$.

$$\Rightarrow \quad \nu_l = - A_{lj} g^{jk} dF(\partial_k) = - A^k_l dF(\partial_k).$$

$$\therefore \; I = -2f \, g^{il} g^{kj} F^{\gamma}_{k} h_{\gamma\beta} A_{ij} \Big(- \overbrace{A^{p}_{l}}^{\nu^{\beta}_{l}.} F^{\beta}_{p} \Big)$$

$$= 2f \, \cancel{g^{il}} \, \cancel{g^{kj}} \, \underbrace{\big(F^{\gamma}_{k} F^{\beta}_{p} h_{\gamma\beta} \big)}_{\cancel{g_{kp}}} \; \underbrace{\cancel{A_{ij}} \, \cancel{A^{p}_{l}} \cdot \cancel{\delta^{j}_{p}}}_{A^{l}_{j} \, \cancel{A_{ij}} \, A^{j}_{l}}$$

$$= 2f \, A^{l}_{j} A^{j}_{l} = 2f \, |A|^2 .$$

$$II = g^{ij} \partial_t A_{ij} = g^{ij} \partial_t \langle \tilde{\nabla}_{dF(\partial_i)} dF(\partial_j), \nu \rangle .$$

$$= g^{ij} \langle \tilde{\nabla}_{dF(\partial_t)} \tilde{\nabla}_{dF(\partial_i)} dF(\partial_j), \nu \rangle$$

$$+ g^{ij} \langle \tilde{\nabla}_{dF(\partial_i)} dF(\partial_j), \underline{\tilde{\nabla}_{dF(\partial_t)} \nu} \rangle \quad \in dF(TM)$$

$$= g^{ij} \tilde{R}(dF(\partial_t), dF(\partial_i), dF(\partial_j), \nu)$$

$$+ g^{ij} \langle \tilde{\nabla}_{dF(\partial_i)} \tilde{\nabla}_{dF(\partial_t)} dF(\partial_j), \nu \rangle .$$

normal coordinate at p

$$+ g^{ij} \langle dF(\cancel{\nabla_{\partial_i} \partial_j}), \tilde{\nabla}_{dF(\partial_t)} \nu \rangle .$$

$$= f \, g^{ij} \tilde{R}(\nu, dF(\partial_i), dF(\partial_j), \nu)$$

$$\downarrow\left([\partial_t,\partial_j]=0\right)$$

$$+\ g^{ij}\left\langle \tilde{\nabla}_{dF(\partial_i)}\tilde{\nabla}_{dF(\partial_j)}dF(\partial_t),\ \nu\right\rangle$$

$$=\ f\cdot\tilde{R}_{i\nu}(\nu,\nu)\qquad\left(\because\ \{dF(\partial_i)\}_{i=1}^{n}\cup\{\nu\}\text{ form a basis}\right)$$

$$+\ g^{ij}\left\langle \tilde{\nabla}_{dF(\partial_i)}\tilde{\nabla}_{dF(\partial_j)}(f\nu),\ \nu\right\rangle.$$

$$\Downarrow$$

$$\frac{d^2}{dt^2}\Big|_{t=0}\text{Area}=\int_M -2f^2|A|^2-f^2\tilde{R}_{\nu\nu}$$

$$-\ g^{ij}\left\langle \tilde{\nabla}_{dF(\partial_i)}\tilde{\nabla}_{dF(\partial_j)}(f\nu),\ f\nu\right\rangle d\mu_0$$

where. $$\tilde{\nabla}_{dF(\partial_i)}\tilde{\nabla}_{dF(\partial_j)}(f\nu)=\tilde{\nabla}_{dF(\partial_i)}\left(\partial_j f\cdot\nu+f\,\tilde{\nabla}_{dF(\partial_j)}\nu\right)$$

$$=\partial_i\partial_j f\cdot\nu+\partial_i f\cdot\tilde{\nabla}_{\partial_j}\nu+\partial_j f\,\tilde{\nabla}_{\partial_i}\nu+f\,\tilde{\nabla}_{\partial_i}\tilde{\nabla}_{\partial_j}\nu.$$

$$\overset{\text{Normal coord}}{=}\ \nabla_i\nabla_j f\cdot\nu+f_i\cancel{\nu_j}+f_j\cancel{\nu_i}+f\,\nabla_i\nabla_j\nu.$$

after inner product with $f\nu$.

$$=\int_M -2f^2|A|^2-f^2\tilde{R}_{\nu\nu}-f\Delta f-f^2g^{ij}\langle\nu_{ij},\nu\rangle.$$

$$\overset{\text{div. thm}}{=}\int_M -2f^2|A|^2-f^2\tilde{R}_{\nu\nu}+|\nabla f|^2+f^2|\nabla\nu|^2\,d\mu_0$$

$$\left(\begin{array}{l} \text{Since} \quad \nabla_i \nu = A_i^l \, dF(\partial_l) \\ \quad |\nabla \nu|^2 = g^{ij} A_i^l A_j^k g_{kl} = |A|^2 \end{array} \right)$$

and hence

$$= \int_M - f^2 \left(|A|^2 + \widetilde{Ric}_{\nu\nu} \right) + |\nabla f|^2 \, d\mu_0 \ldots$$

$$\left(\begin{array}{c} \| \\ \frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(M, g(t)). \end{array} \right)$$

Thm (Simon) If M^n is cpt, minimal in N^{n+1} with $\widetilde{Ric}(N) > 0$, then M is Not stable minimizer.

pf: Otherwise, $\forall f \in C^\infty(M)$,

$$0 \le \frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(M, g(t)) = \int_M |\nabla f|^2 - f^2 \left(\tilde{R}_{\nu\nu} + |A|^2 \right)$$

Put $f = 1$

$$\overset{(\because \widetilde{Ric} > 0)}{\Rightarrow} \quad 0 < \int \tilde{R}_{\nu\nu} + |A|^2 \, d\mu_0 \le 0.$$

$\rightarrow\leftarrow$ #